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Singular Perturbation and Time Scale Approaches in Discrete Control Systems

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Introduction

THE theory of singular perturbations and time scales has been a powerful analytical tool in the analysis and synthesis of continuous and discrete control systems.[1,2] In this Note, we first consider a singularly perturbed discrete control system. Using a singular perturbation approach, outer and correction subsystems are obtained. Next, by the application of a time scale approach via block diagonalization transformations, the original system is decoupled into slow and fast subsystems. It will be shown that, to a zeroth-order approximation, the singular perturation and time scale approaches yield equivalent results. Roughly speaking, the zeroth-order approximation is sometimes called the first approximation. This result is similar to a corresponding result in continuous control systems.[3]

Singular Perturbation Approach

Consider a general form for linear, shift-invariant, singularly perturbed discrete systems as[2]

$$x(k+1)=A_{11}x(k)+h^{1-j}A_{12}z(k)+B_1u(k) \tag{1a}$$

$$h^{2i}z(k+1)=h^jA_{21}x(k)+hA_{22}z(k)+h^jB_2u(k) \tag{1b}$$

$$0\le i\le 1;\ 0\le j\le 1$$

where $x(k)$ and $z(k)$ are "slow" and "fast" state vectors of n and m dimensions, respectively, $u(k)$ an r-dimensional control vector, h a singular perturbation parameter, and A and B matrices of appropriate dimensionality. We formulate initial value problems with $x(k=0)=x(0)$ and $z(k=0)=z(0)$ and note that similar results can be obtained for boundary value problems as well.

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The three limiting cases of Eq. (1) result in the following models:

1) The C-model ($i=0, j=0$),

$$x(k+1)=A_{11}x(k)+hA_{12}z(k)+B_1u(k) \tag{2a}$$

$$z(k+1)=A_{21}x(k)+hA_{22}z(k)+B_2u(k) \tag{2b}$$

where the small parameter h appears in the *column* of the system matrix.

2) The R-model ($i=0, j=1$).

$$x(k+1)=A_{11}x(k)+A_{12}z(k)+B_1u(k) \tag{3a}$$

$$z(k+1)=hA_{21}x(k)+hA_{22}z(k)+hB_2u(k) \tag{3b}$$

where the small parameter h appears in the *row* of the system matrix.

3) The D-model ($i=1, j=1$),

$$x(k+1)=A_{11}x(k)+A_{12}z(k)+B_1u(k) \tag{4a}$$

$$hz(k+1)=A_{21}x(k)+A_{22}z(k)+B_2u(k) \tag{4b}$$

where the small parameter h is positioned in an identical fashion to that of the continuous systems described by *differential* equations. In this Note, we consider only the C-model of Eq. (2), but the result can be extended to the other two models of Eqs. (3) and (4) as well. The outer (degenerate) subsystem, obtained by zeroth-order approximation (i.e., by making $h=0$) of Eq. (2), is

$$x^{(o)}(k+1)=A_{11}x^{(o)}(k)+B_1u^{(o)}(k) \tag{5a}$$

$$z^{(o)}(k+1)=A_{21}x^{(o)}(k)+B_2u^{(o)}(k) \tag{5b}$$

$$x^{(o)}(k=0)=x(0);\ z^{(o)}(k=0)\neq z(0) \tag{5c}$$

Here, we note that in the process of degeneration, $x(k)$ has retained its initial condition $x(0)$, whereas $z(k)$ has lost its initial condition $z(0)$. In order to recover this lost initial condition, a correction subsystem is used.[2] The transformations between the original and correction variables are

$$x_c(k)=x(k)/h^{k+1};\ z_c(k)=z(k)/h^k \tag{6a}$$

$$u_c(k)=u(k)/h^{k+1} \tag{6b}$$

Using Eq. (6) in Eq. (2), the transformed system becomes

$$hx_c(k+1)=A_{11}x_c(k)+A_{12}z_c(k)+B_1u_c(k) \tag{7a}$$

$$z_c(k+1)=A_{21}x_c(k)+A_{22}z_c(k)+B_2u_c(k) \tag{7b}$$

The zeroth-order approximation ($h=0$) of Eq. (7) becomes

$$0=A_{11}x_c{}^{(o)}(k)+A_{12}z_c{}^{(o)}(k)+B_1u_c{}^{(o)}(k) \tag{8a}$$

$$z_c{}^{(o)}(k+1)=A_{21}x_c{}^{(o)}(k)+A_{22}z_c{}^{(o)}(k)+B_2u^{(o)}(k) \tag{8b}$$

Rewriting Eq. (8), we get

$$x_c{}^{(o)}(k)=-A_{11}{}^{-1}[A_{12}z_c{}^{(o)}(k)+B_1u_c{}^{(o)}(k)] \tag{9a}$$

$$z_c{}^{(o)}(k+1)=A_{c0}z_c{}^{(o)}(k)+B_{c0}u_c{}^{(o)}(k) \tag{9b}$$

where

$$A_{c0}=A_{22}-A_{21}A_{11}{}^{-1}A_{12}$$

$$B_{c0}=B_2-A_{21}A_{11}{}^{-1}B_1$$

The total solution consists of an outer solution and a correction solution as[2]

$$x(k) = [x^{(o)}(k) + hx^{(1)}(k) + \ldots] + h^{k+1}[x_c^{(o)}(k) + x_c^{(1)}(k) + \ldots] \tag{10a}$$

$$z(k) = [z^{(o)}(k) + hz^{(1)}(k) + \ldots] + h^{k}[z_c^{(o)}(k) + z_c^{(1)}(k) + \ldots] \tag{10b}$$

For the present, to simplify the analysis, we omit $u(k)$ and its associated functions. Then, for zeroth-order approximation, the total solution is given by[2]

$$x(k) = x^{(o)}(k) \tag{11a}$$

$$z(k) = z^{(o)}(k) + h^k z_c^{(o)}(k) \tag{11b}$$

$$= z^{(o)}(k) + z_r^{(o)}(k) \tag{11c}$$

where, $z_r^{(o)}(k) = h^k z_c^{(o)}(k)$. From Eq. (5c), we note that only $z(k)$ has lost its initial condition. Hence, Eq. (11) gives

$$z_c^{(o)}(k=0) = z(0) - z^{(o)}(0) \tag{12}$$

Our current interest is only zeroth-order approximations. Thus, from Eqs. (5) and (9), we get

$$x^{(o)}(k+1) = A_{11}x^{(o)}(k) \tag{13a}$$

$$z^{(o)}(k+1) = A_{21}A_{11}^{-1}x^{(o)}(k+1) \tag{13b}$$

or

$$z^{(o)}(k) = A_{21}A_{11}^{-1}x^{(o)}(k) \tag{13c}$$

and the correction functions as

$$z_c^{(o)}(k+1) = A_{c0}a_c^{(o)}(k) \tag{14a}$$

or

$$z_r^{(o)}(k+1) = hA_{c0}z_r^{(o)}(k) \tag{14b}$$

where

$$z_r^{(o)}(k=0) = a_c^{(o)}(0) = z(0) - z^{(o)}(0) = z(0) - A_{21}A_{11}^{-1}x(0)$$

Time Scale Approach

Let us consider again the singularly perturbed system of Eq. (2). We now use the time scale approach and obtain slow and fast subsystems to a zeroth-order approximation.

For decoupling the original system of Eq. (2) into slow and fast subsystems, the block diagonalization transformations relating the decoupled variables in terms of the original variables are[4,5]

$$x_s(k) = (I_s + hED)x(k) + hEz(k) \tag{15a}$$

$$z_f(k) = Dx(k) + I_f z(k) \tag{15b}$$

and transformations relating the original variables and the decoupled variables are

$$x(k) = x_s(k) - hEz_f(k) \tag{16a}$$

$$z(k) = -Dx_s(k) + (I_f + hDE)z_f(k) \tag{16b}$$

where $I_s(n\times n)$ and $I_f(m\times m)$ are unity matrices and $D(m\times n)$ and $E(n\times m)$ satisfy Riccati-type algebraic equations.

$$hA_{22}D - DA_{11} + hDA_{12}D - A_{21} = 0 \tag{17a}$$

$$hE(A_{22} + DA_{12}) - (A_{11} - hA_{12}D)E + A_{12} = 0 \tag{17b}$$

whose iterative solutions start with initial values of $D_i = -A_{21}A_{11}^{-1}$ and $E_i = A_{11}^{-1}A_{12}$. By using transformations given by Eq. (15) in Eq. (2), we get the decoupled slow and fast subsystems as,

$$x_s(k+1) = A_s x_s(k) + B_s u(k) \tag{18a}$$

$$z_f(k+1) = hA_f z_f(k) + B_f u(k) \tag{18b}$$

where

$$A_s = A_{11} - hA_{12}D;\ A_f = A_{22} + DA_{12}$$

$$B_s = (I_s + hED)B_1 + hEB_2$$

$$B_f = DB_1 + B_2$$

For zeroth-order approximation,[3] we get

$$D_0 = -A_{21}A_{11}^{-1};\ E_0 = A_{11}^{-1}A_{12}^{-1} \tag{19a}$$

$$A_{s0} = A_{11};\ A_{f0} = A_{22} - A_{21}A_{11}^{-1}A_{12} \tag{19b}$$

$$B_{s0} = B_1;\ B_{f0} = B_2 - A_{21}A_{11}^{-1}B_1 \tag{19c}$$

Using Eq. (19) in Eqs. (16) and (18), we get (omitting input for simplicity)

$$x(k) = x_s^{(o)}(k) \tag{20a}$$

$$z(k) = A_{21}A_{11}^{-1}x_s^{(o)}(k) + z_f^{(o)}(k) \tag{20b}$$

where $x_s^{(o)}(k)$ and $z_f^{(o)}(k)$ safisty

$$x_s^{(o)}(k+1) = A_{11}x_s^{(o)}(k) \tag{21a}$$

$$z_f^{(o)}(k+1) = hA_{fo}z_f^{(o)}(k) \tag{21b}$$

Similarly, using Eq. (19) in Eq. (15), we obtain

$$x_s^{(o)}(k=0) = x(0) \tag{22a}$$

$$z_f^{(o)}(k=0) = z(0) - A_{21}A_{11}^{-1}x(0) \tag{22b}$$

Comparing the subsystems of Eqs. (13) and (14) and the solution of Eq. (11) obtained by using the singular perturbation approach with the corresponding subsystems of Eq. (21) and the solution of Eq. (20), we find that they satisfy the same equations with the same initial conditions. Hence,

$$x^{(o)}(k) = x_s^{(o)}(k);\ z^{(o)}(k) = A_{21}A_{11}^{-1}x_s^{(o)}(k) \tag{23a}$$

$$z_r^{(o)}(k) = z_f^{(o)}(k);\ A_{c0} = A_{f0} \tag{23b}$$

Thus, we have shown that for a zeroth-order approximation, both singular perturbation and time scale approaches give identical results. Similar results can be established for other types of discrete systems characterized by Eqs. (3) and (4).

Conclusion

In this Note, we have demonstrated for a zeroth-order approximation the equivalence of the subsystems obtained by the singular perturbation and time scale approaches. This result is akin to that in the singularly perturbed continuous systems. It can be shown that such an equivalence exists for a first-order approximation also, the details of which are omitted due to the lengthy and cumbersome nature of the derivations.

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Extensional Oscillations of Tethered Satellite Systems

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Introduction

RECENTLY, there has been considerable interest in the use of tethers in space. This has led to many investigations of their dynamics and control, but with a few exceptions, they usually deal with the rotational motion of the tether. However, during normal operations of tethered satellite systems, elastic oscillations of the tether, both transverse and longitudinal, are likely to be excited. The objective of this Note is to discuss some of the issues associated with longitudinal (extensional) oscillations of the tether. It may be pointed out that the resulting oscillations in tension may affect the attitude dynamics of the subsatellite significantly.

In their analysis of longitudinal oscillations of the tether, Banerjee and Kane[1] had assumed that the entire tether, including the undeployed part, undergoes extensional oscillations. In other words, friction between the tether and the reel, as well as that between different layers of the tether, was assumed to be negligible. This is termed the "total-slip" case in this Note. On the other hand, Misra and Modi,[2–3] Kohler et al.,[4] and Bergamaschi et al.[5] considered the other extreme case, in which it is assumed that there is sufficient friction so that the undeployed part of the tether does not undergo any extensional oscillation. This is referred to as the "no-slip" case here. The real case, of course, is somewhere in between. The present Note compares the results for the two extreme cases as well as those for intermediate levels of friction. It is shown that, even for very small friction, the results are quite close to those for the no-slip case. In addition, an analytical solution is presented for the no-slip case, while the previous investigators obtained the solutions numerically.

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Formulation of the Problem

The system under consideration consists of the main orbiter A, subsatellite B, and the tether C (Fig. 1). It is assumed that the Shuttle is in a circular orbit having an orbital angular velocity Ω. A part of the tether is deployed, while the rest – the undeployed part – is wound around the tether reel. Only the longitudinal oscillations of the tether are considered. The reason for doing this is that the main objective of this work is to determine the role played by the undeployed part of the tether; however, the transverse vibrations are not affected by the undeployed part.

The longitudinal elastic deformation or stretch undergone by an element of the tether at any instant t is denoted by $v(s,t)$. Here, the coordinate s describes the position of the element measured along the "unstretched" tether from the point of its attachment to the reel, i.e., s is a material coordinate. The portion of the tether corresponding to $0<s<\ell_w$ is wrapped around the reel, while that corresponding to $\ell_w<s<\ell_t$ is the deployed part. Total deployed length is $\ell_d=\ell_t-\ell_w$.

Using the extended Hamilton's principle[6] (or otherwise), one can obtain the following equation of motion:

$$EA\frac{\partial^2 v}{\partial s^2}-\rho\frac{\partial^2 v}{\partial t^2}+3\rho\Omega^2[v+(s-\ell_w)]\,U(s-\ell_w)$$
$$+[1-U(s-\ell_w)]\{F+3\rho\Omega^2[(a/2)\sin(2s/a)]$$
$$+v\cos^2(s/a)]\}=0 \tag{1}$$

with the boundary conditions

$$v(0,t)=0 \tag{2a}$$

and

$$-EA\frac{\partial v}{\partial s}(\ell_t,t)-m_b\frac{\partial^2 v}{\partial t^2}(\ell_t,t)+3m_b\Omega^2[(\ell_t-\ell_w)$$
$$+v(\ell_t,t)]=0 \tag{2b}$$

Here, E and ρ stand for Young's modulus and mass per unit length of the tether, A the area of its cross section, a the radius of the tether reel, m_b the mass of the subsatellite, F the frictional force per unit length of the wrapped tether and a function of s, while $U(s-\ell_w)$ is the unit step function. In Eq. (1), the first two terms correspond to the classical string vibration problem, the third term is the gravity-gradient force corresponding to the deployed part of the tether, and the last term represents the friction and gravity-gradient force on the wrapped portion of the tether.

Equation (1) in conjunction with the boundary conditions (2) are now analyzed for the three cases of no slip, total slip, and intermediate slip, as described earlier.

Analysis and Results

Analysis for the No-Slip Case

In this case, the undeployed part is prevented from oscillating due to the presence of sufficient friction. Thus, $v(s,t)=0$ for $0<s<\ell_w$ and we need to consider only the

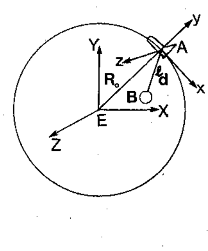

Fig. 1 Tethered Satellite System.